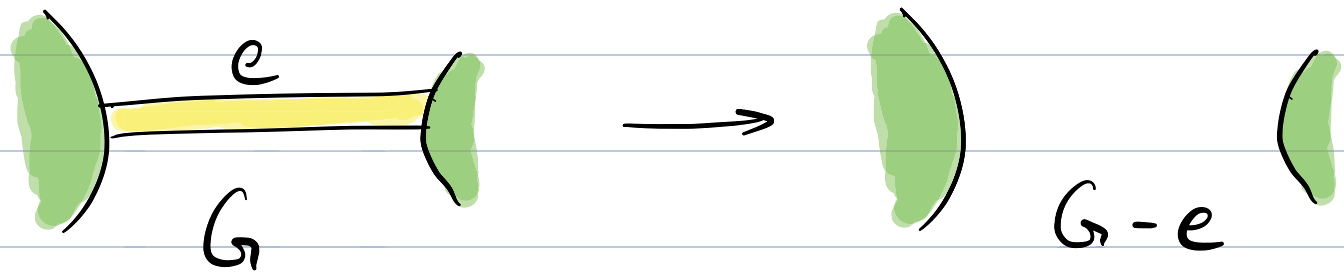
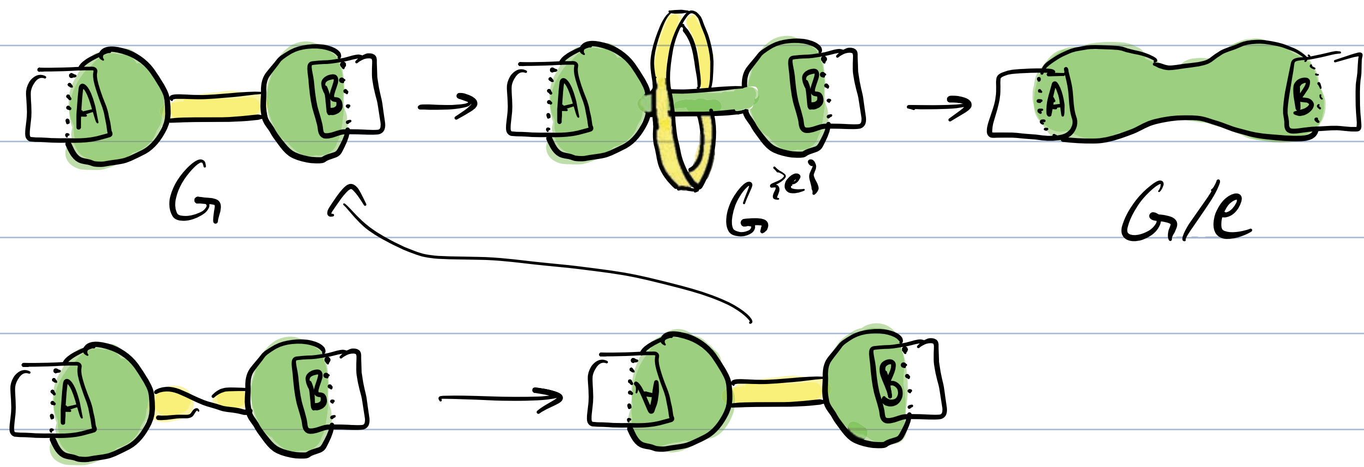
Contraction-Deletion Relations for the Bollobás-Riordan Polynomial

Def: For a ribbon graph G, $e \in E(G)$, A <u>deletion</u> $G-e$ is obtained by removing e from G.



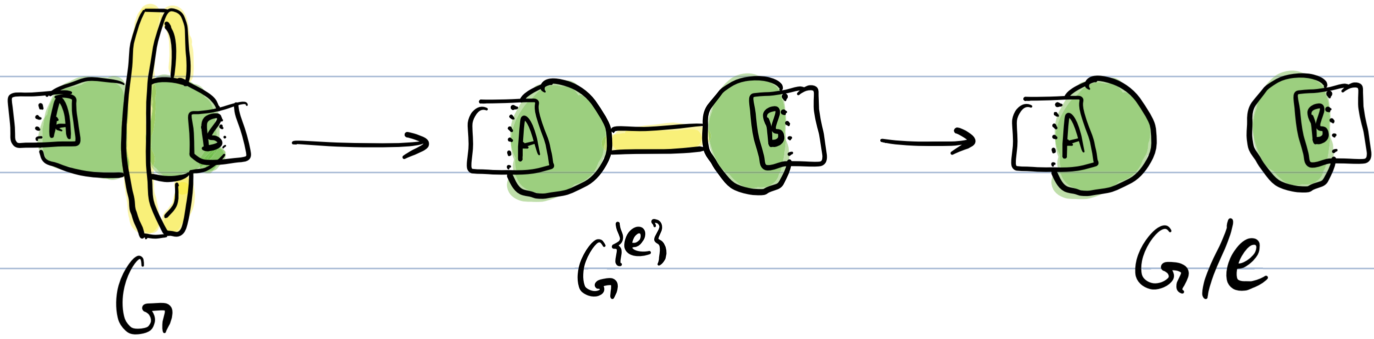
A <u>Contraction</u> $G/e := G^{\{e\}} - e$ where $G^{\{e\}}$ refers to the Partial dual of G w.r.t e.

(i) e is not a loop:

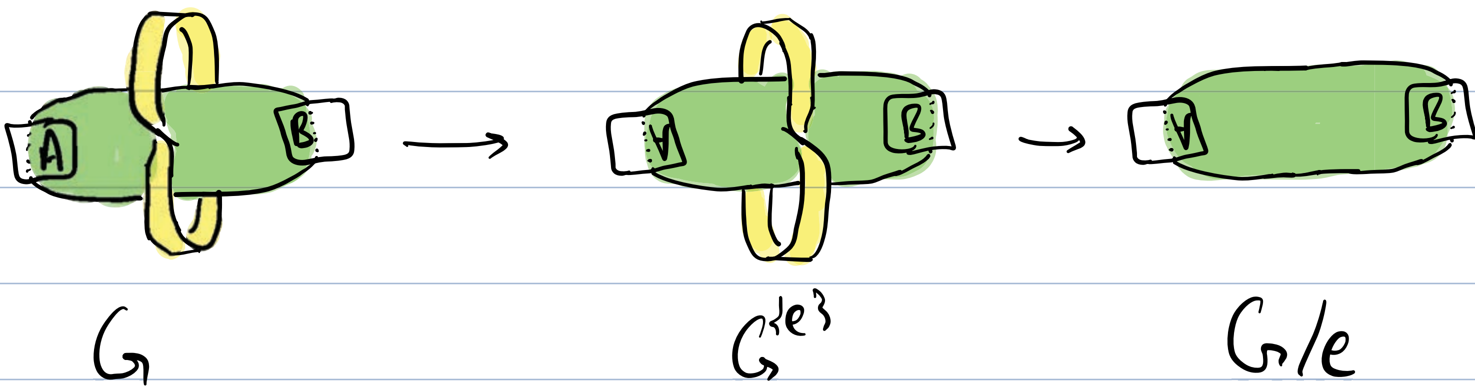


twist the left side

(ii) e is an orientable loop



(iii) e is a non-orientable loop



Signed Bollobás–Riordan Polynomial:

$$R_G(X,Y,Z) = \sum_{F \subseteq E(G)} \Big(\prod_{e \in F} x_e\Big)\Big(\prod_{e \notin F} y_e\Big) X^{r(G)-r(F)}\, Y^{n(F)}\, Z^{k(F)-bc(F)+n(F)}$$

where:

$x_e = \begin{cases} +1 & \text{if } e \text{ is positive} \\ \sqrt{\frac{X}{Y}} & \text{if } e \text{ is negative} \end{cases}$ | $y_e = \begin{cases} +1 & \text{if } e \text{ is positive} \\ \sqrt{\frac{Y}{X}} & \text{if } e \text{ is negative} \end{cases}$

$v(G) = \#(\text{vertices in } G)$,

$e(G) = \#(\text{edges in } G)$

$k(G) = \#(\text{Connected components in } G)$

$r(G) = v(G) - k(G)$

$n(G) = e(G) - r(G)$

$bc(G) = \#(\text{boundary components of } G)$

Def: $e \in E(G)$ is called a bridge if $G - e$ has more connected components than G.

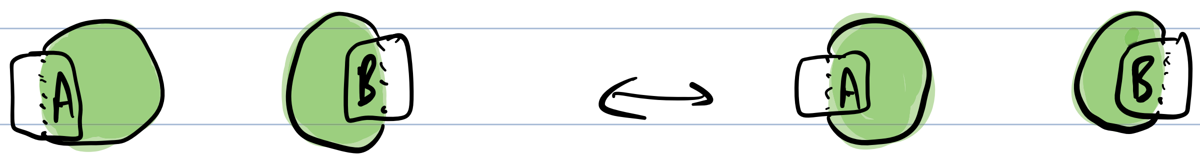
Contraction - deletion properties:

For a signed ribbon graph G, positive edge e of G, we have:

$$R_G = \begin{cases} R_{G/e} + R_{G-e}, & \text{if } e \text{ is ordinary- not a bridge or loop} \\ (x+1) R_{G/e}, & \text{if } e \text{ is a bridge} \end{cases}$$

If e is not a bridge:

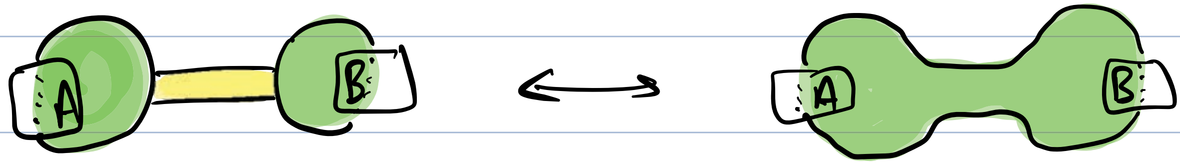
Spanning subgraphs of G that don't contain e are in one to one correspondence with subgraphs of $G - e$.



$F_0 \subseteq E(G) \setminus \{e\}$ $\qquad\qquad$ $F_1 \subseteq E(G-e)$

This correspondence preserves $v(F), e(F), k(F), \ldots$

Similarly there is a one to one correspondence between spanning subgraphs of G containing e and spanning subgraphs of G/e



$F_0 = F \cup \{e\} \subseteq E(G)$ $\qquad\qquad$ $F_1 \subseteq G/e$

we get the following:

$v(F_0) = v(F_1) + 1$, $v(G) = v(G/e) + 1$

$k(F_0) = k(F_1)$, $k(G) = k(G/e)$

$\Rightarrow r(G) - r(F_0) = (r(G/e) + \cancel{1}) - (r(F_1) + \cancel{1})$

$e(F_0) = e(F_1) + 1$

$\Rightarrow n(F_0) = e(F_0) - r(F_0) = (e(F_1) + 1) - (r(F_1) + 1) = n(F_1)$

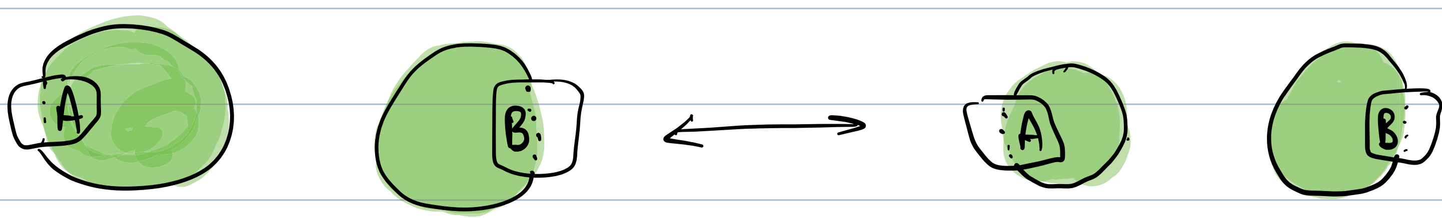
$bc(F_0) = bc(F_1)$

$\rightarrow k(F_0) - bc(F_0) + n(F_0) = k(F_1) - bc(F_1) + n(F_1)$.

$\prod_{e \in F_0} x_e = \prod_{e \in F_1} x_e$, $\prod_{e \notin F_0} y_e = \prod_{e \notin F_1} y_e$

In the case that e is a bridge we get:

$R_G = x R_{G-e} + R_{G/e}$

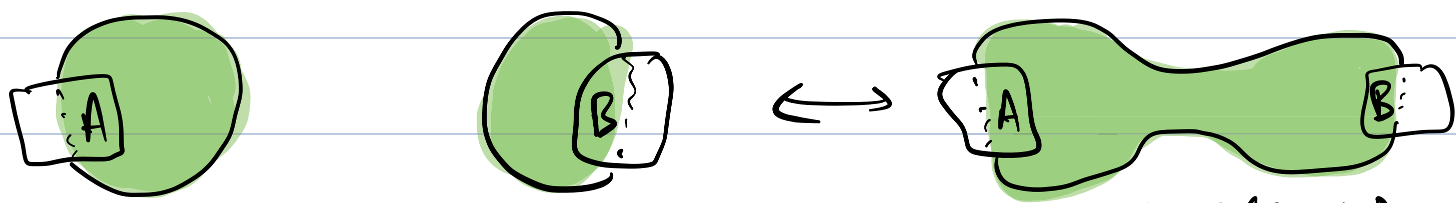


$F_0 \subseteq E(G) \setminus \{e\}$ $\qquad\qquad$ $F_1 \subseteq E(G-e)$

we get:

$$v(G) = v(G-e),\ v(F_0) = v(F_1)$$
$$k(G) = k(G-e) - 1,\ k(F_0) = k(F_1)$$
$$r(G) - r(F_0) = r(G-e) - r(F_1) + 1$$
$$n(F_0) = n(F_1),\ k(F_0) - bc(F_0) + n(F_0) = k(F_1) - bc(F_1) + n(F_1)$$
$$\prod_{e \in F_0} x_e = \prod_{e \in F_1} x_e,\ \prod_{e \notin F_0} y_e = \prod_{e \notin F_1} y_e$$

we also get $R(G-e) = R(G/e)$ due to the following correspondence



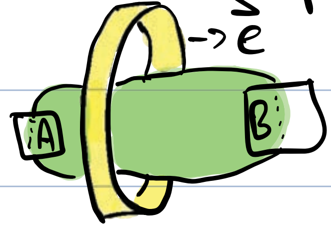
$F_0 \subseteq E(G-e)$ $\qquad$ $F_1 \subseteq E(G/e)$

we can easily check $r(G-e) - r(F_0) = r(G/e) - r(F_1)$
and $n(F_0) = n(F_1)$, $k(G-e) = k(G/e) + 1$
we also get $bc(G-e) = bc(G/e) + 1$

If e is a negative edge, we get:

$$R_G = \begin{cases} \sqrt{\frac{X}{Y}}\, R_{G-e} + \sqrt{\frac{Y}{X}}\, R_{G/e}, & \text{if } e \text{ is ordinary} \\ \sqrt{\frac{X}{Y}}\,(Y+1)\, R_{G/e}, & \text{if } e \text{ is a bridge} \end{cases}$$

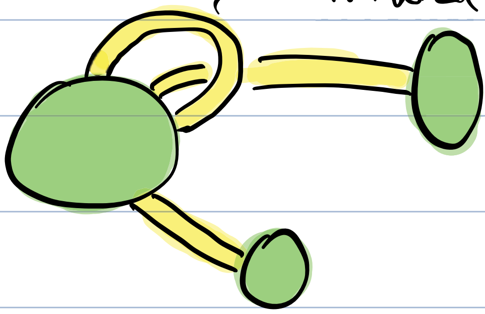
Def: A loop e of G is said to be trivial if in the following picture,



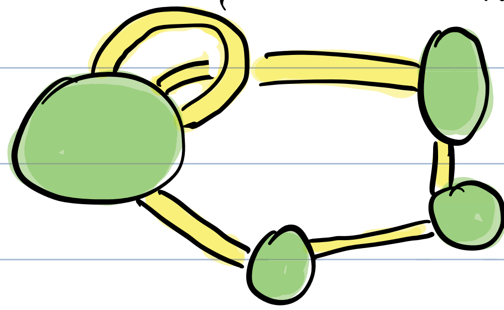
there is no path from A to B that avoids the vertex that e is connected to.

(I drew an orientable loop, same definition also holds if e is not orientable)

trivial loop



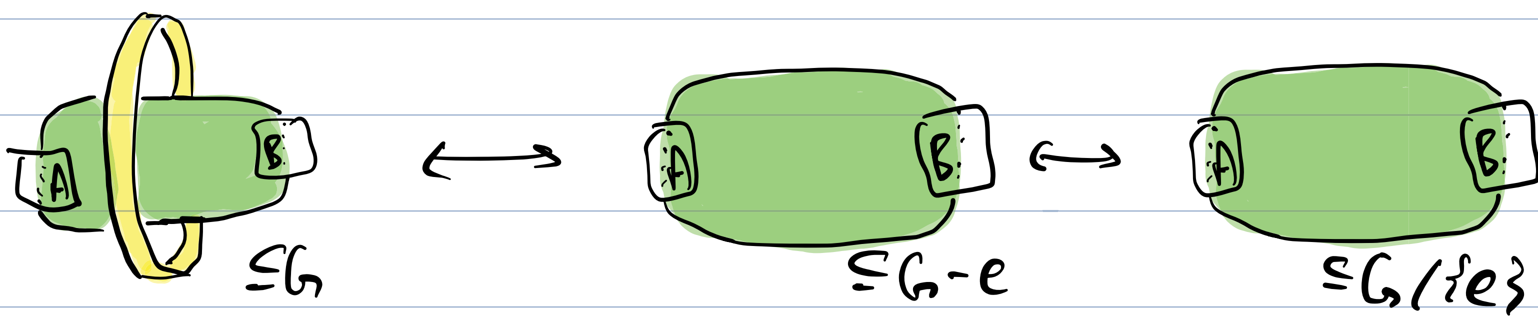
non-trivial loop



we have the following properties for trivial loops:

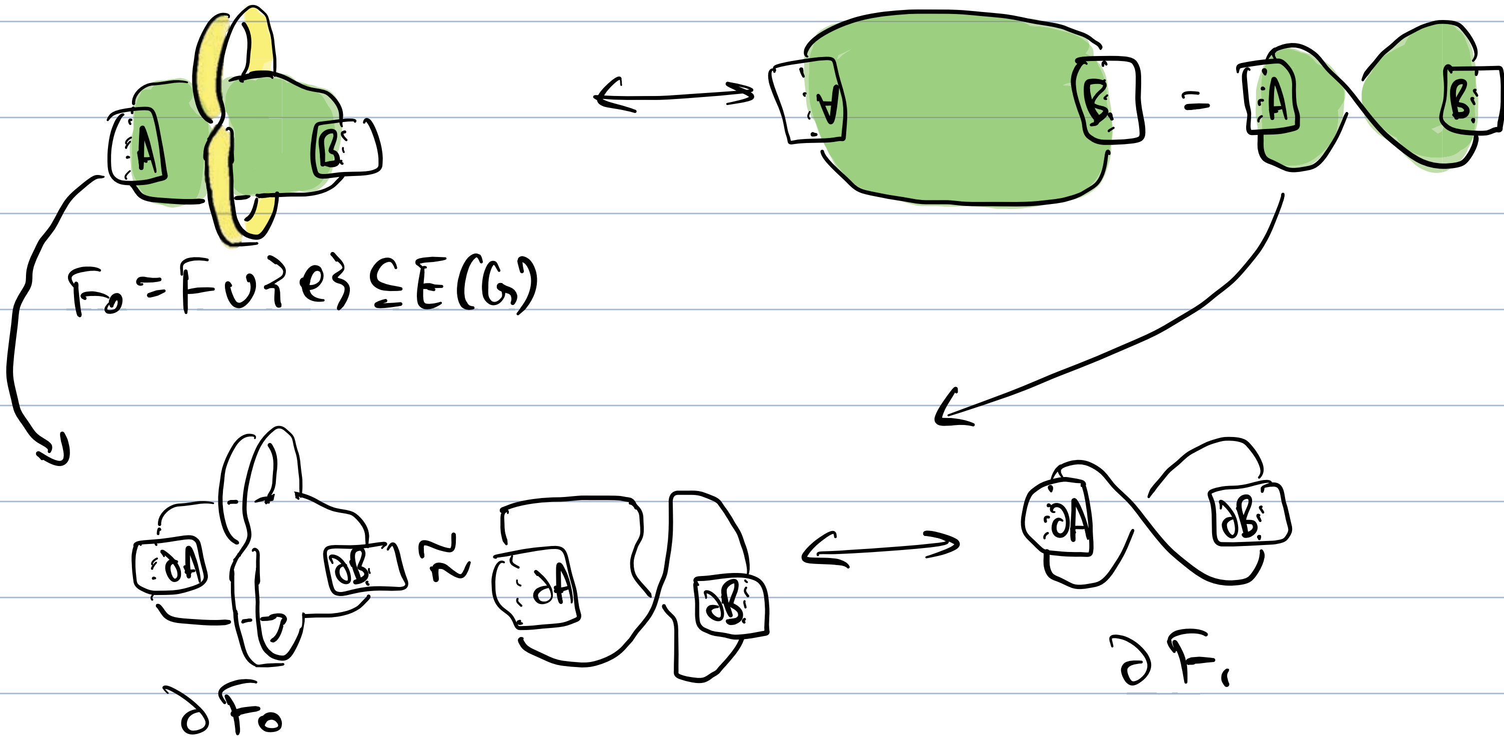
Let e be a trivial loop

$$R_G = \begin{cases} (y+1)\,R_{G-e}, & \text{if } e \text{ is orientable positive} \\ \sqrt{\frac{x}{y}}\,(x+1)R_{G-e}, & \text{if } e \text{ is a orientable negative} \\ (yz+1)\,R_{G-e}, & \text{if } e \text{ is a non orientable positive} \\ \sqrt{\frac{y}{x}}\,(xz+1)\,R_{G-e}, & \text{if } e \text{ is a non orientable negative} \end{cases}$$

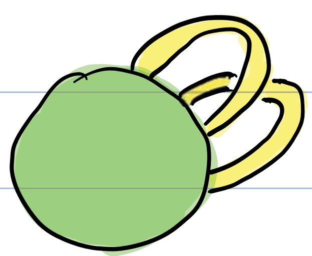


If e is non-orientable loop we have:

$$R_G = \begin{cases} R_{G-e} + yz R_{G/e}, & \text{if } e \text{ is positive} \\ \sqrt{\frac{r}{x}}\left(R_{G-e} + xz R_{G/e}\right), & \text{if } e \text{ is negative} \end{cases}$$

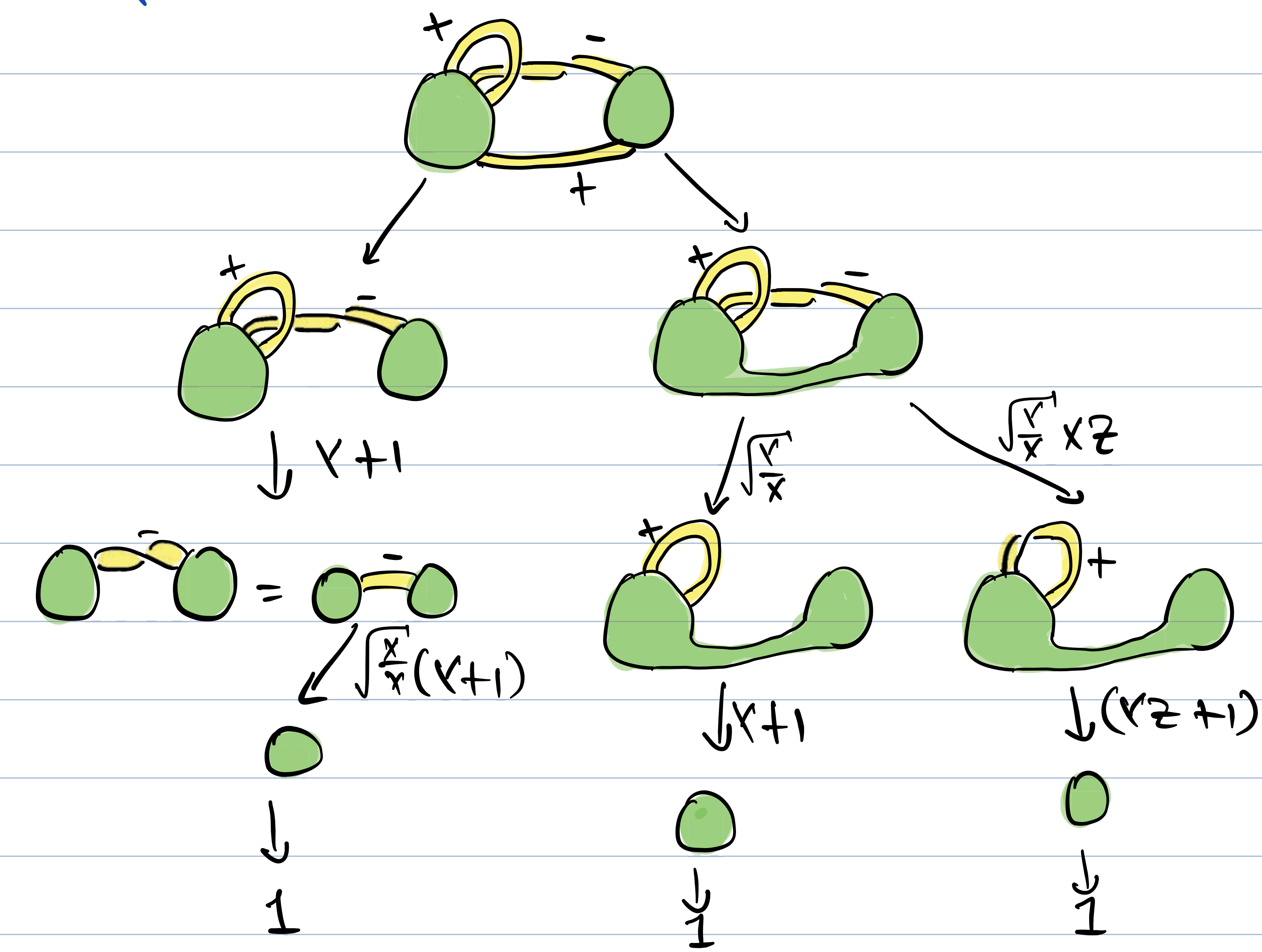


These formulas don't define R_G!



– no way to resolve using relations described above.

Example :



$$\sqrt{\frac{x}{y}}(y+1)^2+\sqrt{\frac{y}{x}}(y+1)+\sqrt{\frac{y}{x}}(xz)(yz+1) = R_G$$

Reference :

Chmutov, Sergei. "Generalized duality for graphs on surfaces and the signed Bollobás–Riordan polynomial." *Journal of Combinatorial Theory, Series B* 99.3 (2009): 617-638.